Corrigé type : Rattrapage math 01.

le 23/06/2023.

Département de physique

Ex 01 : $f : \mathbb{R}-\{1\} \longrightarrow \mathbb{R}$ an application defined by

$$f(x) = \frac{2x-1}{x-1}$$

① f is injective : let $x_1, x_2 \in \mathbb{R}-\{1\}$, we have

$$f(x_1) = f(x_2) \Rightarrow \frac{2x_1-1}{x_1-1} = \frac{2x_2-1}{x_2-1} \Rightarrow (2x_1-1)(x_2-1) = (2x_2-1)(x_1-1)$$

$$\Rightarrow 2x_1x_2 - 2x_1 - x_2 + 1 = 2x_1x_2 + 2x_2 - x_1 \quad 1{,}5$$

$$\Rightarrow x_1 = x_2 \Rightarrow f \text{ is injective.}$$

② f is not surjective : $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}-\{1\} : y = f(x)$?

we have : $y = f(x) \Rightarrow y = \frac{2x-1}{x-1} \Rightarrow y(x-1) = 2x-1$.

$$\Rightarrow yx - y = 2x - 1 \Rightarrow x(y-2) = y - 1 . \quad 01$$

$$\Rightarrow x = \frac{y-1}{y-2}$$

than for $y = 2$ there is not $x \in \mathbb{R}$: $\Rightarrow$ f is not surjective.

③ $g : \mathbb{R}-\{1\} \longrightarrow \mathbb{R}-\{2\}$ by $g(x) = f(x)$

g is surjective since : $\forall x \in \mathbb{R}-\{2\}$ there exist $x \in \mathbb{R}-\{1\}$ and g is injective than g is bijective. and $\boxed{g^{-1}(x) = \frac{x-1}{x-2}}$ 1,5

p.

Ex 01: part 2:

$$h(x) = \begin{cases} x^2 \ln|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) $\lim_{x \to 0} h(x) = \lim_{x \to 0} x^2 \ln|x| = 0$. (famous limit) and $f(0) = 0$ 01

Then f is continous at $x_0 = 0$.

(b) $\lim_{x \overset{>}{\to} 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \ln|x| - 0}{x - 0} = \lim_{x \to 0} \frac{x^2 \ln x}{x} = \lim_{x \to 0} x \ln x = 0$ 01

and $\lim_{x \overset{<}{\to} 0} \frac{h(x) - h(0)}{x - 0} = \lim \frac{x^2 \ln(-x)}{x} = \lim_{x \to 0} x\, h(-x) = 0$

Then f is derivable at $x_0 = 0$

(c) the equation $x^2 \ln x = 1$ has at least a solution in the interval $]1, e^2[$ since the function 01

$f(x) = x^2 \ln x - 1$ is continuous and $h(1) = -1$

and $h(e^2) = e^2 \ln e^2 - 1 = 2e^2 - 1$

$h(1) \cdot h(e^2) < 0$: (using the V.I.T)

Ex 02:

$$U = \{(x, y, z) \in \mathbb{R}^3 : x + y = z\}$$

(1) U is a vector subspace since:

$u + v \in U$.

$\forall u(x, y, z) \in U, \forall v(x', y', z') \in U \Rightarrow$ 0.5

$u(x, y, z) \in U \Rightarrow x + y = z$ and $v \in U \Rightarrow x' + y' = z'$

then $x + x' + y + y' = z + z' \Rightarrow u + v \in U$

and for $\alpha \in \mathbb{R}$ and $u \in U \Rightarrow \alpha u \in U$ since 0.5

$\alpha x + \alpha y = \alpha z$

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① basis of U: $U = \{(x, y, z) \in \mathbb{R}^3 : x + y = z\}$.

$= \{(x, y, z) \in \mathbb{R}^3 ; x = z - y\}$ 0,1

$= \{(z - y, y, z)\} = \{(z, 0, z) + (-y, y, 0) : y, z \in \mathbb{R}\}$

$= \{z\underbrace{(1, 0, 1)}_{v_1} + y\underbrace{(-1, 1, 0)}_{v_2} : y, z \in \mathbb{R}\}$ 0,5

v_1, v_2 are independent. 0,5

part 2 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$; $f(x, y) = (x - y, x + 5y)$

① f is linear: since ; $\forall u, v \in \mathbb{R}^2$; $f(u + v) = f(u) + f(v)$

we have

$f(u + v) = f(x + x', y + y') = (x + x' - y - y', x + x' + 3y + 3y')$ 0,5

$= (x - y, x + 3y) + (x' - y', x' + 3y') = f(u) + f(v)$

and $f(\alpha u) = \alpha f(u)$ since: 0,5

$f(\alpha x, \alpha y) = (\alpha x - \alpha y, \alpha x + 5\alpha y) = \alpha(x - y, x + 5y) = \alpha f(u)$

② $\ker f = \{u(x, y) \in \mathbb{R}^2 ; f(u) = 0_{\mathbb{R}^2}\}$ 0,5

$\Rightarrow \begin{cases} x - y = 0 \\ x + 5y = 0 \end{cases} \Rightarrow x = y = 0 \Rightarrow \ker f = \{0, 0\}$.

③ since $\ker f = \{0, 0\} \Rightarrow f$ is injective. 0,5

and $\dim \ker f + \dim \operatorname{Im} f = \dim \mathbb{R}^2 \Rightarrow \dim \operatorname{Im} f = 2$

$\Rightarrow f$ is surjective. 0,5

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Ex 03: 1) $\forall x, y \in \mathbb{R}$: $x R y \iff x^2 - y^2 = x - y$.

R is Reflexive since $x R x \iff x^2 - x^2 = x - x$ is verify. 01

R is symetric since $x R y \Rightarrow x^2 - y^2 = x - y$

$\Rightarrow y^2 - x^2 = y - x \Rightarrow y R x$ 01

R is transitive since: $\forall x, y, z \in \mathbb{R}$.

$x R y \Rightarrow x^2 - y^2 = x - y \quad \text{--} (1)$

$y R z \Rightarrow y^2 - z^2 = y - z \quad \text{--} (2)$ $\Rightarrow x^2 - z^2 = x - z \Rightarrow x R z$

01.

(2) Equivalent class of 1:

$\dot{1} = \{x \in \mathbb{R} ; x^2 - 1^2 = x - 1\} = \{x \in \mathbb{R} ; x^2 - x = 0\}$ 01

then $\dot{1} = \{0, 1\}$.

$\dot{\alpha} = \{x \in \mathbb{R} ; x^2 - \alpha^2 = x - \alpha\} = \{x \in \mathbb{R} ; x^2 - x - \alpha^2 + \alpha = 0\}$

$\Delta = 1 - 4(-\alpha^2 + \alpha) = 1 + 4\alpha^2 + 4\alpha = (2\alpha + 1)^2$

$x_1 = \dfrac{1 + 2\alpha + 1}{2} = 1 + \alpha \qquad x_2 = \dfrac{1 - 2\alpha - 1}{2} = -\alpha$ 01.

$\dot{\alpha} = \{1 + \alpha, -\alpha\}$

Exo 4: ① $f : U \to V$ and $g : V \to W$ are linear map:

ker f is subspace: since: $u \in \ker f \Rightarrow f(u) = 0_V$.
$v \in \ker f \Rightarrow f(v) = 0_V$ (01).

and $f(u) + f(v) = f(u+v) = 0_V \Rightarrow u + v \in \ker f$.

and $f(\alpha u) = \alpha f(u) = 0_V \Rightarrow \alpha u \in \ker f$ (0,5)

$\Rightarrow$ ker f is subspace of U (0,5)

f linear.

② let $u, v \in U \Rightarrow g(f(u+v)) = g(f(u) + g(v))$ (01).

$= g(f(u)) + g(f(v))$ since g is linear.

then $g \circ f(u+v) = g \circ f(u) + g \circ f(v)$ (0,5)

and $g(f(\alpha u)) = g(\alpha f(u)) = \alpha g(f(u)) \Rightarrow$ then $g \circ f$ is linear:

③ $g \circ f(U) = g(f(U)) = g(V) = W$ (0,5)

then $g \circ f$ is surjective

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